

Ladder operators and differential equations for multiple orthogonal polynomials

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Abstract

In this paper, we obtain the ladder operators and associated compatibility conditions for the type I and the type II multiple orthogonal polynomials. These ladder equations extend known results for orthogonal polynomials and can be used to derive the differential equations satisfied by multiple orthogonal polynomials. Our approach is based on Riemann-Hilbert problems and the Christoffel-Darboux formula for multiple orthogonal polynomials, and the nearest-neighbor recurrence relations. As an illustration, we give several explicit examples involving multiple Hermite and Laguerre polynomials, and multiple orthogonal polynomials with exponential weights and cubic potentials.

Keywords: Ladder operators, differential equations, multiple orthogonal polynomials, Riemann-Hilbert problems, nearest-neighbor recurrence relations.

1 Introduction

1.1 Multiple orthogonal polynomials

Multiple orthogonal polynomials are polynomials of one variable which are defined by orthogonality relations with respect to r different measures $\mu_1, \mu_2, \dots, \mu_r$, where $r \geq 1$ is a positive integer. As a generalization of orthogonal polynomials [15, 25, 34], multiple orthogonal polynomials originated from Hermite-Padé approximation in the context of irrationality and transcendence proofs in number theory. They were further developed in approximation theory, cf. [2, 4, 10, 26, 33] and surveys [3, 35, 36]. During the past few years, multiple orthogonal polynomials have also arisen in a natural way in certain models from mathematical physics, including random matrix theory, non-intersecting paths, etc.; we refer to [30, 31] and references therein for the progress of this subject.

Let $\vec{n} = (n_1, n_2, \dots, n_r) \in \mathbb{N}^r$ be a multi-index of size $|\vec{n}| = n_1 + n_2 + \dots + n_r$ and suppose $\mu_1, \mu_2, \dots, \mu_r$ are r measures with supports on certain simple curves in the complex plane. There are two types of multiple orthogonal polynomials. The *type I multiple orthogonal*

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polynomials are given by the vector $(A_{\vec{n},1}, \dots, A_{\vec{n},r})$, where $A_{\vec{n},j}$ is a polynomial of degree $\leq n_j - 1$, for which

$$\sum_{j=1}^r \int x^k A_{\vec{n},j}(x) d\mu_j(x) = 0, \quad k = 0, 1, \dots, |\vec{n}| - 2. \quad (1.1)$$

We use the normalization

$$\sum_{j=1}^r \int x^{|\vec{n}|-1} A_{\vec{n},j}(x) d\mu_j(x) = 1. \quad (1.2)$$

The equations (1.1)–(1.2) are a linear system of $|\vec{n}|$ equations for the unknown coefficients of $A_{\vec{n},1}, \dots, A_{\vec{n},r}$. This system has a unique solution if the matrix of mixed moments

$$M_{\vec{n}} = \left(\begin{array}{cccc|cccc} \nu_0^{(1)} & \nu_1^{(1)} & \cdots & \nu_{n_1-1}^{(1)} & & & & \\ \nu_1^{(1)} & \nu_2^{(1)} & \cdots & \nu_{n_1}^{(1)} & & & & \\ \vdots & \vdots & \cdots & \vdots & & & & \\ \nu_{|\vec{n}|-1}^{(1)} & \nu_{|\vec{n}|}^{(1)} & \cdots & \nu_{|\vec{n}|+n_1-2}^{(1)} & & & & \\ & & & & \cdots & & & \\ & & & & & \nu_0^{(r)} & \nu_1^{(r)} & \cdots & \nu_{n_r-1}^{(r)} \\ & & & & & \nu_1^{(r)} & \nu_2^{(r)} & \cdots & \nu_{n_r}^{(r)} \\ & & & & & \vdots & \vdots & \cdots & \vdots \\ & & & & & \nu_{|\vec{n}|-1}^{(r)} & \nu_{|\vec{n}|}^{(r)} & \cdots & \nu_{|\vec{n}|+n_r-2}^{(r)} \end{array} \right),$$

where

$$\nu_n^{(j)} = \int x^n d\mu_j(x), \quad j = 1, \dots, r,$$

is not singular, in which case we call the multi-index \vec{n} a *normal index*. The *type II multiple orthogonal polynomial* is the monic polynomial $P_{\vec{n}}(x) = x^{|\vec{n}|} + \dots$ of degree $|\vec{n}|$ for which

$$\begin{aligned} \int P_{\vec{n}}(x) x^k d\mu_1(x) &= 0, \quad k = 0, 1, \dots, n_1 - 1, \\ &\vdots \\ \int P_{\vec{n}}(x) x^k d\mu_r(x) &= 0, \quad k = 0, 1, \dots, n_r - 1. \end{aligned} \quad (1.3)$$

This gives a linear system of $|\vec{n}|$ equations for the $|\vec{n}|$ unknown coefficients of $P_{\vec{n}}$ and the matrix of this linear system is the transpose $M_{\vec{n}}^T$ of (1.1), hence $P_{\vec{n}}$ exists and is unique whenever \vec{n} is a normal index.

Suppose that the r measures μ_1, \dots, μ_r are all absolutely continuous with respect to a measure μ and that $d\mu_j(x) = w_j(x) d\mu(x)$. The type I and II multiple orthogonal polynomials satisfy the following biorthogonality:

$$\int P_{\vec{n}}(x) Q_{\vec{m}}(x) d\mu(x) = \begin{cases} 0, & \text{if } \vec{m} \leq \vec{n}, \\ 0, & \text{if } |\vec{n}| \leq |\vec{m}| - 2, \\ 1, & \text{if } |\vec{m}| = |\vec{n}| + 1, \end{cases} \quad (1.4)$$

where

$$Q_{\vec{n}}(x) = \sum_{j=1}^r A_{\vec{n},j}(x) w_j(x), \quad (1.5)$$

see [28, Theorem 23.1.6].

For more information about multiple orthogonal polynomials, we refer to Aptekarev et al. [3, 5], Coussement and Van Assche [38], Nikishin and Sorokin [33, Chapter 4, §3], and Ismail [28, Chapter 23]. Throughout this paper, we shall assume that all multi-indices are normal.

1.2 Recurrence relations

There are several recurrence relations for the type I and the type II multiple orthogonal polynomials. For type II, we have the following nearest-neighbor recurrence relations

$$\begin{aligned} xP_{\vec{n}}(x) &= P_{\vec{n}+\vec{e}_1}(x) + b_{\vec{n},1}P_{\vec{n}}(x) + \sum_{j=1}^r a_{\vec{n},j}P_{\vec{n}-\vec{e}_j}(x), \\ &\vdots \\ xP_{\vec{n}}(x) &= P_{\vec{n}+\vec{e}_r}(x) + b_{\vec{n},r}P_{\vec{n}}(x) + \sum_{j=1}^r a_{\vec{n},j}P_{\vec{n}-\vec{e}_j}(x), \end{aligned} \tag{1.6}$$

where $\vec{e}_j = (0, \dots, 0, 1, 0, \dots, 0)$ is the j -th standard unit vector with 1 on the j -th entry, $(a_{\vec{n},1}, \dots, a_{\vec{n},r})$ and $(b_{\vec{n},1}, \dots, b_{\vec{n},r})$ are the recurrence coefficients. These recurrence relations were derived in [28, Theorem 23.1.11] where it was shown that

$$a_{\vec{n},j} = \frac{\int x^{n_j} P_{\vec{n}}(x) d\mu_j(x)}{\int x^{n_j-1} P_{\vec{n}-\vec{e}_j}(x) d\mu_j(x)}, \tag{1.7}$$

and

$$b_{\vec{n},j} = \int xP_{\vec{n}}(x)Q_{\vec{n}+\vec{e}_j}(x) d\mu(x), \tag{1.8}$$

for $j = 1, \dots, r$, where $Q_{\vec{n}}$ is given in (1.5).

We have similar recurrence relations for the type I multiple orthogonal polynomials:

$$\begin{aligned} xQ_{\vec{n}}(x) &= Q_{\vec{n}-\vec{e}_1}(x) + b_{\vec{n}-\vec{e}_1,1}Q_{\vec{n}}(x) + \sum_{j=1}^r a_{\vec{n},j}Q_{\vec{n}+\vec{e}_j}(x), \\ &\vdots \\ xQ_{\vec{n}}(x) &= Q_{\vec{n}-\vec{e}_r}(x) + b_{\vec{n}-\vec{e}_r,r}Q_{\vec{n}}(x) + \sum_{j=1}^r a_{\vec{n},j}Q_{\vec{n}+\vec{e}_j}(x). \end{aligned} \tag{1.9}$$

Observe that the same recurrence coefficients $(a_{\vec{n},1}, \dots, a_{\vec{n},r})$ are used but that there is a shift in the other recurrence coefficients $(b_{\vec{n}-\vec{e}_1,1}, \dots, b_{\vec{n}-\vec{e}_r,r})$.

If we multiply (1.6) by $Q_{\vec{n}}(x)$ and integrate, then the biorthogonality (1.4) gives

$$\int xP_{\vec{n}}(x)Q_{\vec{n}}(x) d\mu(x) = \sum_{j=1}^r a_{\vec{n},j}.$$

The orthogonality properties of $P_{\vec{n}}$ in (1.3) imply that

$$\int P_{\vec{n}}(x)Q_{\vec{n}+\vec{e}_j}(x) d\mu(x) = \int P_{\vec{n}}(x)A_{\vec{n}+\vec{e}_j,j}(x) d\mu_j(x),$$

hence, by (1.4),

$$1 = \kappa_{\vec{n}+\vec{e}_j,j} \int x^{n_j} P_{\vec{n}}(x) d\mu_j(x), \tag{1.10}$$

where $A_{\vec{n},j}(x) = \kappa_{\vec{n},j}x^{n_j-1} + \dots$. This, together with (1.7), provides an alternative representation of $a_{\vec{n},j}$:

$$a_{\vec{n},j} = \frac{\int x^{n_j} P_{\vec{n}}(x) d\mu_j(x)}{\int x^{n_j-1} P_{\vec{n}-\vec{e}_j}(x) d\mu_j(x)} = \frac{\kappa_{\vec{n},j}}{\kappa_{\vec{n}+\vec{e}_j,j}}. \quad (1.11)$$

1.3 Ladder equations for orthogonal polynomials

Given a single positive measure $w(x)dx$, it is well-known that there exist monic orthogonal polynomials $P_n(x)$ of degree n in x such that

$$\int P_m(x) P_n(x) w(x) dx = h_n \delta_{m,n}, \quad h_n > 0, \quad m, n = 0, 1, 2, \dots \quad (1.12)$$

They are characterized by the three-term recurrence relation:

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x), \quad (1.13)$$

where

$$\alpha_n = \frac{1}{h_n} \int x P_n^2(x) w(x) dx, \quad \beta_n = \frac{1}{h_{n-1}} \int x P_n(x) P_{n-1}(x) w(x) dx, \quad (1.14)$$

and the initial condition is taken to be $\beta_0 P_{-1}(x) := 0$. Suppose that w vanishes at the end points of the orthogonality interval, then it was shown in [11] that P_n satisfy the following ladder equations:

$$\left(\frac{d}{dx} + B_n(x) \right) P_n(x) = \beta_n A_n(x) P_{n-1}(x), \quad (1.15)$$

$$\left(\frac{d}{dx} - B_n(x) - \mathbf{v}'(x) \right) P_{n-1}(x) = -A_{n-1}(x) P_n(x) \quad (1.16)$$

with $\mathbf{v}(x) := -\ln w(x)$ and

$$A_n(x) := \frac{1}{h_n} \int \frac{\mathbf{v}'(x) - \mathbf{v}'(y)}{x - y} [P_n(y)]^2 w(y) dy, \quad (1.17)$$

$$B_n(x) := \frac{1}{h_{n-1}} \int \frac{\mathbf{v}'(x) - \mathbf{v}'(y)}{x - y} P_{n-1}(y) P_n(y) w(y) dy. \quad (1.18)$$

Furthermore, the functions A_n and B_n defined by (1.17) and (1.18) satisfy

$$B_{n+1}(z) + B_n(z) = (z - \alpha_n) A_n(z) - \mathbf{v}'(z), \quad (S_1)$$

$$1 + (z - \alpha_n)[B_{n+1}(z) - B_n(z)] = \beta_{n+1} A_{n+1}(z) - \beta_n A_{n-1}(z). \quad (S_2)$$

The conditions S_1 and S_2 are usually called the compatibility conditions for the ladder equations. Similar relations were also achieved for discrete orthogonal polynomials [29], q -orthogonal polynomials [12] and matrix orthogonal polynomials [20, 27].

The motivation of deriving such ladder equations is two-fold. On the one hand, the differential recurrence relations (1.15) and (1.16), together with recurrence relation (1.13), will lead to a linear differential equation satisfied by P_n . On the other hand, ladder operators has been successfully applied to establish the connections between Painlevé

equations and recurrence coefficients of certain orthogonal polynomials; cf. [6, 9, 13, 14, 19, 21, 22, 23] for recent applications.

It is the aim of this paper to find and explore ladder relations for multiple orthogonal polynomials. We hope our results will be helpful in the further study of recurrence coefficients of multiple orthogonal polynomials and its relation to certain integrable systems. In the literature, the first attempt in this direction was done by Coussement and Van Assche [16, 38], where they derived raising and lowering operators for type II multiple orthogonal polynomials with respect to classical weights. The results there, however, are different from ours, since they consider relations between multiple orthogonal polynomials associated with different weights in the same class.

The results of this paper will be presented in the next section.

2 Statement of the results

2.1 Ladder equations for multiple orthogonal polynomials

Our first result concerns the ladder equations for the type II multiple orthogonal polynomials.

Theorem 2.1. *Let μ_1, \dots, μ_r be r measures that are absolutely continuous with weights w_1, \dots, w_r and each w_i vanishes at the endpoints of the support of μ_i . Suppose that all the indices $\vec{n} = (n_1, \dots, n_r) \in \mathbb{N}^r$ are normal and the functions*

$$\{w_1, xw_1, \dots, x^{n_1-1}w_1, w_2, xw_2, \dots, x^{n_2-1}w_2, \dots, w_r, xw_r, \dots, x^{n_r-1}w_r\}$$

are linearly independent, then we have the following lowering equation for the type II multiple orthogonal polynomials:

$$\begin{aligned} P'_{\vec{n}}(x) = & P_{\vec{n}}(x) \int P_{\vec{n}}(t) \sum_{k=1}^r A_{\vec{n},k}(t) \frac{v'_k(t) - v'_k(x)}{x - t} w_k(t) dt \\ & - \sum_{j=1}^r a_{\vec{n},j} P_{\vec{n}-\vec{e}_j}(x) \int P_{\vec{n}}(t) \sum_{k=1}^r A_{\vec{n}+\vec{e}_j,k}(t) \frac{v'_k(t) - v'_k(x)}{x - t} w_k(t) dt, \end{aligned} \quad (2.1)$$

where $v_k(x) := -\ln w_k(x)$ and $a_{\vec{n},j}$ are the recurrence coefficients given in (1.7).

The following raising equations for the type II multiple orthogonal polynomials hold: for $i = 1, \dots, r$ one has

$$\begin{aligned} P'_{\vec{n}-\vec{e}_i}(x) = & P_{\vec{n}}(x) \int P_{\vec{n}-\vec{e}_i}(t) \sum_{k=1}^r A_{\vec{n},k}(t) \frac{v'_k(t) - v'_k(x)}{x - t} w_k(t) dt \\ & - \sum_{j=1}^r \left(a_{\vec{n},j} \int P_{\vec{n}-\vec{e}_i}(t) \sum_{k=1}^r A_{\vec{n}+\vec{e}_j,k}(t) \frac{v'_k(t) - v'_k(x)}{x - t} w_k(t) dt \right. \\ & \left. - v'_i(x) \delta_{i,j} \right) P_{\vec{n}-\vec{e}_j}(x), \end{aligned} \quad (2.2)$$

where $\delta_{i,j}$ is the Kronecker delta.

Although our assumption on the weight functions may seem a little bit involved, they include the so-called Angelesco systems and AT systems that are normal for all indices; cf. [28, Sections 23.1.1–23.1.2] for an introduction of these two weight systems. Moreover, in the case of $r = 1$, suppose $d\mu_1(x) = w(x)dx$, we have, in the notations of multiple orthogonal polynomials introduced in Sections 1.1 and 1.2,

$$P_{\vec{n}}(x) = P_n(x), \quad A_{\vec{n},1}(x) = \frac{P_{n-1}(x)}{h_{n-1}}, \quad b_{\vec{n},1} = \alpha_n, \quad a_{\vec{n},1} = \beta_n,$$

where P_n , h_n , α_n and β_n are given in (1.12) and (1.14). Inserting the above representations into (2.1) and (2.2), we recover equations (1.15) and (1.16) by noting the fact that $\beta_n = h_n/h_{n-1}$.

The results in Theorem 2.1 can be summarized in vector form. To this end, we introduce an $(r+1) \times (r+1)$ matrix

$$N(x) := N(\vec{n}; x) = (N_{ij}(x))_{0 \leq i, j \leq r}, \quad (2.3)$$

where

$$N_{ij}(x) = \begin{cases} \int P_{\vec{n}-\vec{e}_i}(t) \sum_{k=1}^r A_{\vec{n},k}(t) \frac{v'_k(t) - v'_k(x)}{x-t} w_k(t) dt, & j = 0, \\ a_{\vec{n},j} \int P_{\vec{n}-\vec{e}_i}(t) \sum_{k=1}^r A_{\vec{n}+\vec{e}_j,k}(t) \frac{v'_k(t) - v'_k(x)}{t-x} w_k(t) dt + v'_i(x) \delta_{i,j}, & j \neq 0. \end{cases} \quad (2.4)$$

Here, it is understood that $\vec{e}_0 = \vec{0}$. We then have from (2.1) and (2.2) that

$$\mathbf{P}'_{\vec{n}}(x) = N(x) \mathbf{P}_{\vec{n}}(x), \quad (2.5)$$

where

$$\mathbf{P}_{\vec{n}}(x) := (P_{\vec{n}}(x), P_{\vec{n}-\vec{e}_1}(x), \dots, P_{\vec{n}-\vec{e}_r}(x))^T. \quad (2.6)$$

We have similar results for the type I multiple orthogonal polynomials.

Theorem 2.2. *Let μ_1, \dots, μ_r be r measures as given in Theorem 2.1. For $l = 1, \dots, r$, the associated type I multiple orthogonal polynomials satisfy the following raising equations:*

$$\begin{aligned} A'_{\vec{n},l}(x) = & -A_{\vec{n},l}(x) \int P_{\vec{n}}(t) \sum_{k=1}^r A_{\vec{n},k}(t) \frac{v'_k(t) - v'_k(x)}{x-t} w_k(t) dt \\ & + \sum_{j=1}^r a_{\vec{n},j} A_{\vec{n}+\vec{e}_j,l}(x) \int P_{\vec{n}-\vec{e}_j}(t) \sum_{k=1}^r A_{\vec{n},k}(t) \frac{v'_k(t) - v'_k(x)}{x-t} w_k(t) dt, \end{aligned} \quad (2.7)$$

where $v_k(x) := -\ln w_k(x)$ and $a_{\vec{n},j}$ are the recurrence coefficients given in (1.7).

The lowering equations for the type I multiple orthogonal polynomials are given by

$$\begin{aligned} A'_{\vec{n}+\vec{e}_i,l}(x) = & -A_{\vec{n},l}(x) \int P_{\vec{n}}(t) \sum_{k=1}^r A_{\vec{n}+\vec{e}_i,k}(t) \frac{v'_k(t) - v'_k(x)}{x-t} w_k(t) dt \\ & + \sum_{j=1}^r \left(a_{\vec{n},j} \int P_{\vec{n}-\vec{e}_j}(t) \sum_{k=1}^r A_{\vec{n}+\vec{e}_i,k}(t) \frac{v'_k(t) - v'_k(x)}{x-t} w_k(t) dt \right. \\ & \left. - v'_i(x) \delta_{i,j} \right) A_{\vec{n}+\vec{e}_j,l}(x), \end{aligned} \quad (2.8)$$

for $i = 1, \dots, r$.

In vector form, Theorem 2.2 reads

$$\frac{d}{dx} \begin{pmatrix} -A_{\vec{n},l}(x) \\ a_{\vec{n},1}A_{\vec{n}+\vec{e}_1,l}(x) \\ \vdots \\ a_{\vec{n},r}A_{\vec{n}+\vec{e}_r,l}(x) \end{pmatrix} = N^T(x) \begin{pmatrix} A_{\vec{n},l}(x) \\ -a_{\vec{n},1}A_{\vec{n}+\vec{e}_1,l}(x) \\ \vdots \\ -a_{\vec{n},r}A_{\vec{n}+\vec{e}_r,l}(x) \end{pmatrix}, \quad l = 1, \dots, r. \quad (2.9)$$

From the lowering and raising equations stated in Theorems 2.1 and 2.2, one readily derives differential equations of order $r+1$ for the type II and the type I multiple orthogonal polynomials. As an illustration, we show how to derive differential equations satisfied by $P_{\vec{n}}(x)$ with $r=2$, the idea of which can, of course, be extended to general situations. With $r=2$ in (2.6), i.e., $\mathbf{P}_{\vec{n}}(x) = (P_{\vec{n}}(x), P_{\vec{n}-\vec{e}_1}(x), P_{\vec{n}-\vec{e}_2}(x))^T$, it follows from (2.5) and straightforward calculations that

$$\begin{aligned} \mathbf{P}'_{\vec{n}}(x) &= N(x)\mathbf{P}_{\vec{n}}(x), \\ \mathbf{P}''_{\vec{n}}(x) &= N_1(x)\mathbf{P}_{\vec{n}}(x), \\ \mathbf{P}'''_{\vec{n}}(x) &= N_2(x)\mathbf{P}_{\vec{n}}(x), \end{aligned}$$

where

$$\begin{aligned} N_1(x) &= N'(x) + N^2(x), \\ N_2(x) &= N''(x) + 2N'(x)N(x) + N(x)N'(x) + N^3(x), \end{aligned}$$

and the derivative of a matrix valued function is understood in the entry-size manner. Hence,

$$P'_{\vec{n}}(x) = (N)_{00}(x)P_{\vec{n}}(x) + (N)_{01}(x)P_{\vec{n}-\vec{e}_1}(x) + (N)_{02}(x)P_{\vec{n}-\vec{e}_2}(x), \quad (2.10)$$

$$P''_{\vec{n}}(x) = (N_1)_{00}(x)P_{\vec{n}}(x) + (N_1)_{01}(x)P_{\vec{n}-\vec{e}_1}(x) + (N_1)_{02}(x)P_{\vec{n}-\vec{e}_2}(x), \quad (2.11)$$

$$P'''_{\vec{n}}(x) = (N_2)_{00}(x)P_{\vec{n}}(x) + (N_2)_{01}(x)P_{\vec{n}-\vec{e}_1}(x) + (N_2)_{02}(x)P_{\vec{n}-\vec{e}_2}(x), \quad (2.12)$$

where we use the notation $(M)_{ij}$ to denote the (i, j) -th entry of any given matrix M . Then, we can represent $P_{\vec{n}-\vec{e}_1}$ and $P_{\vec{n}-\vec{e}_2}$ in terms of $P_{\vec{n}}$, $P'_{\vec{n}}$ and $P''_{\vec{n}}$ by solving (2.10) and (2.11). Finally, replacing $P_{\vec{n}-\vec{e}_1}$ and $P_{\vec{n}-\vec{e}_2}$ in (2.12) by these relations will lead us to the third order differential equations satisfied by $P_{\vec{n}}$. Since the exact formulas are cumbersome, we plan not to write them down here, but will present some concrete examples in Section 5.

2.2 Compatibility conditions

We finally state the compatibility conditions for the ladder equations for multiple orthogonal polynomials. First we write the nearest-neighbor recurrence relations in a vector form. Let $\mathbf{P}_{\vec{n}}(x)$ be defined by (2.6), then we have

$$\mathbf{P}_{\vec{n}+\vec{e}_l}(x) = W(\vec{n} + \vec{e}_l; x)\mathbf{P}_{\vec{n}}(x), \quad (2.13)$$

for $l = 1, \dots, r$, where

$$W(\vec{n} + \vec{e}_l; x) = \begin{pmatrix} x - b_{\vec{n},l} & -a_{\vec{n},1} & -a_{\vec{n},2} & \cdots & -a_{\vec{n},l} & \cdots & -a_{\vec{n},r} \\ 1 & B_{l,1}(\vec{n}) & 0 & \cdots & 0 & \cdots & 0 \\ 1 & 0 & B_{l,2}(\vec{n}) & & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \ddots & \\ 1 & 0 & 0 & \cdots & 0 & 0 & B_{l,r}(\vec{n}) \end{pmatrix} \quad (2.14)$$

is a matrix polynomial of degree 1 with

$$B_{l,j}(\vec{n}) = b_{\vec{n}-\vec{e}_j,j} - b_{\vec{n}-\vec{e}_j,l}, \quad j = 1, \dots, r.$$

The formula (2.13) follows from the nearest-neighbor recurrence relations (1.6), since by eliminating $xP_{\vec{n}}(x)$ between l -th and j -th relation in (1.6), we have

$$P_{\vec{n}+\vec{e}_l}(x) - P_{\vec{n}+\vec{e}_j}(x) = (b_{\vec{n},j} - b_{\vec{n},l})P_{\vec{n}}(x).$$

Rewriting these relations and the l -th relation in (1.6) gives us (2.13) and (2.14). We then have

Theorem 2.3. *The compatibility conditions for the ladder equations stated in Theorem 2.1 are given by*

$$N(\vec{n} + \vec{e}_l; x)W(\vec{n} + \vec{e}_l; x) = W'(\vec{n} + \vec{e}_l; x) + W(\vec{n} + \vec{e}_l; x)N(\vec{n}; x), \quad l = 1, \dots, r, \quad (2.15)$$

where $N(\vec{n}; x)$ is defined in (2.5).

Clearly, by considering (2.15) in an entry-size manner, one has at least $r(r+1)^2$ equalities, which are much more complicated than the single weight case. Similar compatibility conditions for ladder equations associated with the type I multiple orthogonal polynomials can be obtained easily by using (2.15). We omit the results here.

2.3 Outline of the paper

The rest of this paper is mainly devoted to the proofs of our theorems. We will prove Theorems 2.1 and 2.2 in Section 3. Unlike the treatment of the single weight case [11], Riemann-Hilbert (RH) problems and the Christoffel-Darboux formula for multiple orthogonal polynomials will be two fundamental components in the derivations of the ladder equations; see also [27] for similar treatment to matrix orthogonal polynomials. We then derive the compatibility conditions in Section 4, where we also give a review of partial difference equations satisfied by the nearest-neighbor recurrence coefficients, obtained in [37], for later use. We conclude this paper by applying our results to several concrete examples, namely, multiple Hermite and Laguerre polynomials, and multiple orthogonal polynomials with exponential weights and cubic potentials.

3 Proofs of Theorems 2.1 and 2.2

As mentioned before, the proofs of Theorems 2.1 and 2.2 rely on the Riemann-Hilbert (RH) problem and the Christoffel-Darboux formula for multiple orthogonal polynomials. In what follows, we first give a brief introduction of these two aspects.

3.1 Riemann-Hilbert problems and the Christoffel-Darboux formula for multiple orthogonal polynomials

The usual orthogonal polynomials can be characterized by a RH problem of size 2×2 [24]. It was shown by Van Assche et al. [39] that multiple orthogonal polynomials can also be described in terms of a RH problem, but for matrices of order $(r+1) \times (r+1)$.

For convenience, we assume that all the measures are supported on an oriented, unbounded and simple curve Γ in the complex plane. If the weight functions w_j are Hölder continuous, we look for an $(r+1) \times (r+1)$ matrix valued function Y satisfying the following RH problem:

1. Y is analytic on $\mathbb{C} \setminus \Gamma$.
2. Y possesses continuous boundary values $Y_+(x)$ (from the positive side of Γ) and $Y_-(x)$ (from the negative side of Γ), which satisfies

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_1(x) & w_2(x) & \cdots & w_r(x) \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \quad x \in \Gamma.$$

3. As $z \rightarrow \infty$, $z \in \mathbb{C} \setminus \Gamma$, we have

$$Y(z) = (I + \mathcal{O}(1/z)) \text{diag}(z^{|\vec{n}|}, z^{-n_1}, z^{-n_2}, \dots, z^{-n_r}).$$

If the indices \vec{n} , $\vec{n} - \vec{e}_j$, $j = 1, \dots, r$, are normal, then there exists a unique solution in terms of the type II multiple orthogonal polynomials which is given by

$$Y = \begin{pmatrix} P_{\vec{n}}(z) & \frac{1}{2\pi i} \int_{\Gamma} \frac{P_{\vec{n}}(t)w_1(t)}{t-z} dt & \cdots & \frac{1}{2\pi i} \int_{\Gamma} \frac{P_{\vec{n}}(t)w_r(t)}{t-z} dt \\ -2\pi i \gamma_{\vec{n},1} P_{\vec{n}-\vec{e}_1}(z) & -\gamma_{\vec{n},1} \int_{\Gamma} \frac{P_{\vec{n}-\vec{e}_1}(t)w_1(t)}{t-z} dt & \cdots & -\gamma_{\vec{n},1} \int_{\Gamma} \frac{P_{\vec{n}-\vec{e}_1}(t)w_r(t)}{t-z} dt \\ \vdots & \vdots & \cdots & \vdots \\ -2\pi i \gamma_{\vec{n},r} P_{\vec{n}-\vec{e}_r}(z) & -\gamma_{\vec{n},r} \int_{\Gamma} \frac{P_{\vec{n}-\vec{e}_r}(t)w_1(t)}{t-z} dt & \cdots & -\gamma_{\vec{n},r} \int_{\Gamma} \frac{P_{\vec{n}-\vec{e}_r}(t)w_r(t)}{t-z} dt \end{pmatrix}, \quad (3.1)$$

where

$$\frac{1}{\gamma_{\vec{n},j}} = \int_{\Gamma} x^{n_j-1} P_{\vec{n}-\vec{e}_j}(t) w_j(t) dt, \quad j = 1, \dots, r. \quad (3.2)$$

There exists a similar RH problem for the type I multiple orthogonal polynomials to determine a matrix valued function X of dimension $(r+1) \times (r+1)$ such that

1. X is analytic on $\mathbb{C} \setminus \Gamma$.

2. For $x \in \Gamma$, we have

$$X_+(x) = X_-(x) \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -w_1(x) & 1 & 0 & \cdots & 0 \\ -w_2(x) & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & 0 \\ -w_r(x) & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

3. As $z \rightarrow \infty$, $z \in \mathbb{C} \setminus \Gamma$, we have

$$X(z) = (I + \mathcal{O}(1/z)) \operatorname{diag}(z^{-|\vec{n}|}, z^{n_1}, z^{n_2}, \dots, z^{n_r}).$$

Assume that the indices \vec{n} , $\vec{n} - \vec{e}_j$, $j = 1, \dots, r$, are normal, then the unique solution X is given by

$$X = \begin{pmatrix} \int_{\Gamma} \frac{Q_{\vec{n}}(t)}{z-t} dt & 2\pi i A_{\vec{n},1}(z) & \cdots & 2\pi i A_{\vec{n},r}(z) \\ \frac{c_{\vec{n},1}}{2\pi i} \int_{\Gamma} \frac{Q_{\vec{n}+\vec{e}_1}(t)}{z-t} dt & c_{\vec{n},1} A_{\vec{n}+\vec{e}_1,1}(z) & \cdots & c_{\vec{n},1} A_{\vec{n}+\vec{e}_1,r}(z) \\ \vdots & \vdots & \cdots & \vdots \\ \frac{c_{\vec{n},r}}{2\pi i} \int_{\Gamma} \frac{Q_{\vec{n}+\vec{e}_r}(t)}{z-t} dt & c_{\vec{n},r} A_{\vec{n}+\vec{e}_r,1}(z) & \cdots & c_{\vec{n},r} A_{\vec{n}+\vec{e}_r,r}(z) \end{pmatrix}, \quad (3.3)$$

where $Q_{\vec{n}}$ is defined in (1.5) and

$$c_{\vec{n},j} = \frac{1}{\kappa_{\vec{n}+\vec{e}_j,j}}, \quad j = 1, \dots, r, \quad (3.4)$$

with $\kappa_{\vec{n}+\vec{e}_j,j}$ being the leading coefficient of $A_{\vec{n}+\vec{e}_j,j}$.

If the curve Γ is bounded, extra conditions at the endpoints of Γ are required in the RH problem to ensure the uniqueness of the solution, if the solution exists.

A simple relation between the matrix functions X for the type I and Y for the type II multiple orthogonal polynomials is given by Mahler's relation (cf. [28, Theorem 23.8.3]):

$$X(z) = Y^{-T}(z), \quad (3.5)$$

where A^{-T} is the transpose of the inverse of the matrix A . As we shall see later on, this relation plays an important role in our proof.

We conclude this section with the Christoffel-Darboux formula for multiple orthogonal polynomials. This formula was given by Daems and Kuijlaars [17] as stated in the following theorem (see also [37] for another proof):

Theorem 3.1. *Suppose $(\vec{n}_i)_{i=0,1,\dots,|\vec{n}|}$ is a path in \mathbb{N}^r such that $\vec{n}_0 = \vec{0}$, $\vec{n}_{|\vec{n}|} = \vec{n}$ and for every $i \in \{0, 1, \dots, |\vec{n}| - 1\}$ one has $\vec{n}_{i+1} - \vec{n}_i = \vec{e}_k$ for some $k \in \{1, 2, \dots, r\}$. Then*

$$(x-y) \sum_{i=0}^{|\vec{n}|-1} P_{\vec{n}_i}(x) Q_{\vec{n}_{i+1}}(y) = P_{\vec{n}}(x) Q_{\vec{n}}(y) - \sum_{j=1}^r a_{\vec{n},j} P_{\vec{n}-\vec{e}_j}(x) Q_{\vec{n}+\vec{e}_j}(y). \quad (3.6)$$

Observe that the right hand side is independent of the path $(\vec{n}_i)_{i=0,1,\dots,|\vec{n}|}$ in \mathbb{N}^r . A similar formula for multiple orthogonal polynomials of mixed type is derived in [1]. This formula is of importance in the analysis of certain random matrices [7, 30] and non-intersecting Brownian motions [18].

It is worthwhile to point out that if the system of measures is such that all the indices are normal (perfect system) and the functions

$$\{w_1, xw_1, \dots, x^{n_1-1}w_1, w_2, xw_2, \dots, x^{n_2-1}w_2, \dots, w_r, xw_r, \dots, x^{n_r-1}w_r\}$$

are linearly independent (this is exactly our assumption on the r measures in Theorem 2.1), then the Christoffel-Darboux formula also holds in a component-wise manner, i.e.,

$$(x-y) \sum_{i=0}^{|\vec{n}|-1} P_{\vec{n}_i}(x) A_{\vec{n}_{i+1},k}(y) = P_{\vec{n}}(x) A_{\vec{n},k}(y) - \sum_{j=1}^r a_{\vec{n},j} P_{\vec{n}-\vec{e}_j}(x) A_{\vec{n}+\vec{e}_j,k}(y), \quad (3.7)$$

for $y \in \text{supp}(\mu_k)$ and $k = 1, \dots, r$. To see this, we rewrite (3.6) as

$$\sum_{k=1}^r R_k(x, y) w_k(y) = 0,$$

where

$$R_k(x, y) = (x-y) \sum_{i=0}^{|\vec{n}|-1} P_{\vec{n}_i}(x) A_{\vec{n}_{i+1},k}(y) - P_{\vec{n}}(x) A_{\vec{n},k}(y) + \sum_{j=1}^r a_{\vec{n},j} P_{\vec{n}-\vec{e}_j}(x) A_{\vec{n}+\vec{e}_j,k}(y).$$

With x fixed, it is easily seen that each $R_i(x, y)$ is a polynomial in y of degree less or equal to n_k . The linear independence then implies $R_i(x, y) = 0$, which is (3.7).

3.2 Proof of Theorem 2.1

Let us define an $(r+1) \times (r+1)$ matrix M by

$$M(x) = M(\vec{n}; x) = (M_{ij}(x))_{0 \leq i, j \leq r} := Y'(x) Y^{-1}(x), \quad (3.8)$$

or, equivalently,

$$Y'(x) = M(x) Y(x), \quad (3.9)$$

where Y is given by (3.1). Since the first column of Y is given in terms of the type II multiple orthogonal polynomials, it is easily seen that

$$P'_{\vec{n}}(x) = M_{00}(x) P_{\vec{n}}(x) - \sum_{j=1}^r 2\pi i \gamma_{\vec{n},j} M_{0j}(x) P_{\vec{n}-\vec{e}_j}(x), \quad (3.10)$$

and

$$P'_{\vec{n}-\vec{e}_i}(x) = -\frac{1}{2\pi i \gamma_{\vec{n},i}} \left(M_{i0}(x) P_{\vec{n}}(x) - \sum_{j=1}^r 2\pi i \gamma_{\vec{n},j} M_{ij}(x) P_{\vec{n}-\vec{e}_j}(x) \right), \quad (3.11)$$

for $i = 1, \dots, r$.

In the following we use the notation

$$C(f)(x) = \int_{\Gamma} \frac{f(t)}{t-x} dt, \quad x \in \mathbb{C} \setminus \Gamma,$$

for the Cauchy transform of f . The derivative $dC(f)(x)/dx$ is denoted by $C(f)'(x)$.

On the other hand, the Mahler relation (3.5) gives

$$M(x) = Y'(x)X^T(x).$$

It then follows from (3.1) and (3.3) that

$$M_{00}(x) = -P'_{\vec{n}}(x)C(Q_{\vec{n}})(x) + \sum_{k=1}^r A_{\vec{n},k}(x)C(P_{\vec{n}}w_k)'(x), \quad (3.12)$$

$$M_{0j}(x) = \frac{c_{\vec{n},j}}{2\pi i} \left(-P'_{\vec{n}}(x)C(Q_{\vec{n}+\vec{e}_j})(x) + \sum_{k=1}^r A_{\vec{n}+\vec{e}_j,k}(x)C(P_{\vec{n}}w_k)'(x) \right) \quad (3.13)$$

with $j = 1, \dots, r$. For $i, j = 1, \dots, r$, we have

$$M_{i0}(x) = 2\pi i \gamma_{\vec{n},i} \left(P'_{\vec{n}-\vec{e}_i}(x)C(Q_{\vec{n}})(x) - \sum_{k=1}^r A_{\vec{n},k}(x)C(P_{\vec{n}-\vec{e}_i}w_k)'(x) \right), \quad (3.14)$$

$$M_{ij}(x) = \gamma_{\vec{n},i} c_{\vec{n},j} \left(P'_{\vec{n}-\vec{e}_i}(x)C(Q_{\vec{n}+\vec{e}_j})(x) - \sum_{k=1}^r A_{\vec{n}+\vec{e}_j,k}(x)C(P_{\vec{n}-\vec{e}_i}w_k)'(x) \right). \quad (3.15)$$

The entries M_{ij} in M can be simplified in view of the following two elementary facts.

Proposition 3.1. *If*

$$\int_{\Gamma} t^k f(t) dt = 0, \quad k = 0, \dots, n-1,$$

then we have

$$p(x)C(f)(x) = C(pf)(x)$$

for any polynomial p of degree less than or equal to n .

Indeed,

$$C(pf)(x) = \int_{\Gamma} \frac{f(t)(p(t) - p(x))}{t-x} dt + p(x)C(f)(x)$$

and $(p(t) - p(x))/(t-x)$ is a polynomial of degree less than or equal to $n-1$, hence, by assumption, the integral disappears.

Integration by parts gives us the following statement.

Proposition 3.2. *If f is a differentiable and integrable function that vanishes at the endpoints of Γ , one has*

$$C(f)'(x) = C(f')(x).$$

Hence, the orthogonality of P and Q (see (1.3) and (1.4)) implies

$$P'_{\vec{n}}(x)C(Q_{\vec{n}})(x) = C(P'_{\vec{n}}Q_{\vec{n}})(x), \quad (3.16)$$

$$A_{\vec{n},k}(x)C(P_{\vec{n}}w_k)'(x) = C(A_{\vec{n},k}P_{\vec{n}}w_k)'(x) - C(P_{\vec{n}}w_k)(x)A'_{\vec{n},k}(x), \quad k = 1, \dots, r. \quad (3.17)$$

Inserting the above formulas into (3.12), it is readily seen that

$$\begin{aligned} M_{00}(x) &= -C(P'_{\vec{n}}Q_{\vec{n}})(x) + \sum_{k=1}^r (C(A_{\vec{n},k}P_{\vec{n}}w_k)'(x) - C(P_{\vec{n}}w_k)(x)A'_{\vec{n},k}(x)) \\ &= -C(P'_{\vec{n}}Q_{\vec{n}})(x) + C(P_{\vec{n}}Q_{\vec{n}})'(x) - \sum_{k=1}^r C(P_{\vec{n}}w_k)(x)A'_{\vec{n},k}(x) \\ &= C(P_{\vec{n}}Q'_{\vec{n}})(x) - \sum_{k=1}^r C(P_{\vec{n}}w_kA'_{\vec{n},k})(x) \\ &= C\left(P_{\vec{n}}\sum_{k=1}^r A_{\vec{n},k}w'_k\right)(x) = -C\left(P_{\vec{n}}\sum_{k=1}^r A_{\vec{n},k}v'_kw_k\right)(x), \end{aligned} \quad (3.18)$$

where $v_k(x) := -\ln w_k(x)$. In a similar way, we have

$$M_{0j}(x) = -\frac{c_{\vec{n},j}}{2\pi i}C\left(P_{\vec{n}}\sum_{k=1}^r A_{\vec{n}+\vec{e}_j,k}v'_kw_k\right)(x), \quad (3.19)$$

$$M_{i0}(x) = 2\pi i\gamma_{\vec{n},i}C\left(P_{\vec{n}-\vec{e}_i}\sum_{k=1}^r A_{\vec{n},k}v'_kw_k\right)(x). \quad (3.20)$$

The simplification of M_{ij} with $i, j \geq 1$, however, is a little bit tricky. The problem is that $A_{\vec{n}+\vec{e}_j,k}(x)C(P_{\vec{n}-\vec{e}_i}w_k)'(x)$ in (3.15) can not be written in a form similar to (3.17) (indeed, for $i = j = k$ by comparing the degrees of $A_{\vec{n}+\vec{e}_j,k}$ and orthogonality relations (1.3), we see that assumptions of the first proposition are not fulfilled). To overcome this difficulty, recall that $YY^{-1} = YX^T = I$, the (i, j) entry on both sides gives

$$\gamma_{\vec{n},i}c_{\vec{n},j}\left(P_{\vec{n}-\vec{e}_i}(x)C(Q_{\vec{n}+\vec{e}_j})(x) - \sum_{k=1}^r A_{\vec{n}+\vec{e}_j,k}(x)C(P_{\vec{n}-\vec{e}_i}w_k)(x)\right) = \delta_{i,j}, \quad i, j \geq 1.$$

Note that $P_{\vec{n}-\vec{e}_i}C(Q_{\vec{n}+\vec{e}_j}) = C(P_{\vec{n}-\vec{e}_i}Q_{\vec{n}+\vec{e}_j})$. Taking the derivative of the above formula with respect to x , we see

$$\begin{aligned} -\sum_{k=1}^r A_{\vec{n}+\vec{e}_j,k}(x)C(P_{\vec{n}-\vec{e}_i}w_k)'(x) &= \sum_{k=1}^r A'_{\vec{n}+\vec{e}_j,k}(x)C(P_{\vec{n}-\vec{e}_i}w_k)(x) \\ &\quad - C(P'_{\vec{n}-\vec{e}_i}Q_{\vec{n}+\vec{e}_j})(x) - C(P_{\vec{n}-\vec{e}_i}Q'_{\vec{n}+\vec{e}_j})(x). \end{aligned} \quad (3.21)$$

Using (3.21) in (3.15) we have

$$\begin{aligned} M_{ij}(x) &= \gamma_{\vec{n},i}c_{\vec{n},j}\sum_{k=1}^r \left(A'_{\vec{n}+\vec{e}_j,k}(x)C(P_{\vec{n}-\vec{e}_i}w_k)(x) - C(P_{\vec{n}-\vec{e}_i}Q'_{\vec{n}+\vec{e}_j})(x)\right) \\ &= \gamma_{\vec{n},i}c_{\vec{n},j}C\left(\sum_{k=1}^r P_{\vec{n}-\vec{e}_i}A_{\vec{n}+\vec{e}_j,k}v'_kw_k\right)(x), \quad i, j \geq 1. \end{aligned} \quad (3.22)$$

Replacing M_{ij} in (3.10) and (3.11) by (3.18)–(3.20) and (3.22), it follows

$$\begin{aligned} P'_{\vec{n}}(x) &= -C \left(P_{\vec{n}} \sum_{k=1}^r A_{\vec{n},k} v'_k w_k \right) (x) P_{\vec{n}}(x) \\ &\quad + \sum_{j=1}^r a_{\vec{n},j} C \left(P_{\vec{n}} \sum_{k=1}^r A_{\vec{n}+\vec{e}_j,k} v'_k w_k \right) (x) P_{\vec{n}-\vec{e}_j}(x), \end{aligned} \quad (3.23)$$

and

$$\begin{aligned} P'_{\vec{n}-\vec{e}_i}(x) &= -C \left(P_{\vec{n}-\vec{e}_i} \sum_{k=1}^r A_{\vec{n},k}(x) v'_k w_k \right) (x) P_{\vec{n}}(x) \\ &\quad + \sum_{j=1}^r a_{\vec{n},j} C \left(\sum_{k=1}^r P_{\vec{n}-\vec{e}_i} A_{\vec{n}+\vec{e}_j,k} v'_k w_k \right) (x) P_{\vec{n}-\vec{e}_j}(x), \end{aligned} \quad (3.24)$$

for $i = 1, \dots, r$, where we have used the fact that

$$c_{\vec{n},j} \gamma_{\vec{n},j} = a_{\vec{n},j}; \quad (3.25)$$

see (1.11), (3.2), (1.10) and (3.4).

To show (2.1), we observe from (3.23) that

$$\begin{aligned} P'_{\vec{n}}(x) &= P_{\vec{n}}(x) \int_{\Gamma} P_{\vec{n}}(t) \sum_{k=1}^r A_{\vec{n},k}(t) \frac{v'_k(t) - v'_k(x)}{x - t} w_k(t) dt \\ &\quad - \sum_{j=1}^r a_{\vec{n},j} P_{\vec{n}-\vec{e}_j}(x) \int_{\Gamma} P_{\vec{n}}(t) \sum_{k=1}^r A_{\vec{n}+\vec{e}_j,k}(t) \frac{v'_k(t) - v'_k(x)}{x - t} w_k(t) dt + E(x), \end{aligned} \quad (3.26)$$

where

$$E(x) = \sum_{k=1}^r \left(\sum_{j=1}^r a_{\vec{n},j} C(P_{\vec{n}} A_{\vec{n}+\vec{e}_j,k} w_k)(x) P_{\vec{n}-\vec{e}_j}(x) - C(P_{\vec{n}} A_{\vec{n},k} w_k)(x) P_{\vec{n}}(x) \right) v'_k(x). \quad (3.27)$$

It is then equivalent to show $E = 0$. To this end, we apply the Christoffel-Darboux formula (3.7) to each coefficient of v'_k in (3.27) and obtain

$$\begin{aligned} &\sum_{j=1}^r a_{\vec{n},j} C(P_{\vec{n}} A_{\vec{n}+\vec{e}_j,k} w_k)(x) P_{\vec{n}-\vec{e}_j}(x) - C(P_{\vec{n}} A_{\vec{n},k} w_k)(x) P_{\vec{n}}(x) \\ &= \sum_{i=0}^{|\vec{n}|-1} \left(\int_{\Gamma} P_{\vec{n}}(t) A_{\vec{n}_{i+1},k}(t) w_k(t) dt \right) P_{\vec{n}_i}(x) = 0, \end{aligned}$$

where we have used the orthogonality of $P_{\vec{n}}$ (1.3) in the last equality. Hence, it is immediate that $E = 0$ and (2.1) follows.

In a similar way, we see from (3.24) that

$$\begin{aligned} P'_{\vec{n}-\vec{e}_i}(x) &= P_{\vec{n}}(x) \int_{\Gamma} P_{\vec{n}-\vec{e}_i}(t) \sum_{k=1}^r A_{\vec{n},k}(t) \frac{v'_k(t) - v'_k(x)}{x - t} w_k(t) dt \\ &\quad - \sum_{j=1}^r a_{\vec{n},j} P_{\vec{n}-\vec{e}_j}(x) \int_{\Gamma} P_{\vec{n}-\vec{e}_i}(t) \sum_{k=1}^r A_{\vec{n}+\vec{e}_j,k}(t) \frac{v'_k(t) - v'_k(x)}{x - t} w_k(t) dt \\ &\quad + E_i(x), \end{aligned} \quad (3.28)$$

where

$$E_i(x) = \sum_{k=1}^r \left(\sum_{j=1}^r a_{\vec{n},j} C(P_{\vec{n}-\vec{e}_i} A_{\vec{n}+\vec{e}_j,k} w_k)(x) P_{\vec{n}-\vec{e}_j}(x) - C(P_{\vec{n}-\vec{e}_i} A_{\vec{n},k} w_k)(x) P_{\vec{n}}(x) \right) v'_k(x).$$

Again with the aid of the Christoffel-Darboux formula (3.7) and the orthogonality of $P_{\vec{n}-\vec{e}_i}$, it is readily checked that

$$\begin{aligned} & \sum_{j=1}^r a_{\vec{n},j} C(P_{\vec{n}-\vec{e}_i} A_{\vec{n}+\vec{e}_j,k} w_k)(x) P_{\vec{n}-\vec{e}_j}(x) - C(P_{\vec{n}-\vec{e}_i} A_{\vec{n},k} w_k)(x) P_{\vec{n}}(x) \\ &= \sum_{i=0}^{|\vec{n}|-1} \left(\int_{\Gamma} P_{\vec{n}-\vec{e}_i}(t) A_{\vec{n}_{i+1},k}(t) w_k(t) dt \right) P_{\vec{n}_i}(x) = P_{\vec{n}-\vec{e}_i}(x) \delta_{i,j}. \end{aligned}$$

This, together with (3.26), implies

$$E_i(x) = P_{\vec{n}-\vec{e}_i}(x) v'_i(x). \quad (3.29)$$

Combining (3.28) and (3.29) then leads to (2.2). This completes the proof of Theorem 2.1.

Remark. There is a simple relation between the matrices M in (3.8) and N in (2.3). Note that the first column on both sides of (3.9) gives us

$$\frac{d}{dx} \begin{pmatrix} P_{\vec{n}}(x) \\ -2\pi i \gamma_{\vec{n},1} P_{\vec{n}-\vec{e}_1}(x) \\ \vdots \\ -2\pi i \gamma_{\vec{n},r} P_{\vec{n}-\vec{e}_r}(x) \end{pmatrix} = M(x) \begin{pmatrix} P_{\vec{n}}(x) \\ -2\pi i \gamma_{\vec{n},1} P_{\vec{n}-\vec{e}_1}(x) \\ \vdots \\ -2\pi i \gamma_{\vec{n},r} P_{\vec{n}-\vec{e}_r}(x) \end{pmatrix}.$$

We then obtain from (2.5) that

$$N(x) = \text{diag} \left(1, -\frac{1}{2\pi i \gamma_{\vec{n},1}}, \dots, -\frac{1}{2\pi i \gamma_{\vec{n},r}} \right) M(x) \text{diag} (1, -2\pi i \gamma_{\vec{n},1}, \dots, -2\pi i \gamma_{\vec{n},r}). \quad (3.30)$$

3.3 Proof of Theorem 2.2

We shall prove Theorem 2.2 by establishing (2.9). The formulas (2.7) and (2.8) then follow from straightforward calculations using (2.9) and (2.3).

To see (2.9), we first take a derivative with respect to x on both sides of the equation $Y(x)Y^{-1}(x) = I$ and obtain from (3.8) that

$$Y(x)(Y^{-1})'(x) = -M(x).$$

Equivalently, in view of (3.5),

$$(X^T)'(x) = -X^T(x)M(x). \quad (3.31)$$

Next, we observe from (3.3) that, by transposing the l -th ($l = 1, \dots, r$) row on both sides of (3.31), we get

$$\frac{d}{dx} \begin{pmatrix} 2\pi i A_{\vec{n},l}(x) \\ c_{\vec{n},1} A_{\vec{n}+\vec{e}_1,l}(x) \\ \vdots \\ c_{\vec{n},r} A_{\vec{n}+\vec{e}_r,l}(x) \end{pmatrix} = -M^T(x) \begin{pmatrix} 2\pi i A_{\vec{n},l}(x) \\ c_{\vec{n},1} A_{\vec{n}+\vec{e}_1,l}(x) \\ \vdots \\ c_{\vec{n},r} A_{\vec{n}+\vec{e}_r,l}(x) \end{pmatrix},$$

that is,

$$\begin{aligned} \frac{d}{dx} \begin{pmatrix} A_{\vec{n},l}(x) \\ A_{\vec{n}+\vec{e}_1,l}(x) \\ \vdots \\ A_{\vec{n}+\vec{e}_r,l}(x) \end{pmatrix} &= -\text{diag} \left(\frac{1}{2\pi i}, \frac{1}{c_{\vec{n},1}}, \dots, \frac{1}{c_{\vec{n},r}} \right) M^T(x) \text{diag} (2\pi i, c_{\vec{n},1}, \dots, c_{\vec{n},r}) \\ &\quad \times \begin{pmatrix} A_{\vec{n},l}(x) \\ A_{\vec{n}+\vec{e}_1,l}(x) \\ \vdots \\ A_{\vec{n}+\vec{e}_r,l}(x) \end{pmatrix}. \end{aligned}$$

Replacing M^T in the above formula with the aid of (3.30), we finally arrive at (2.9) after simple manipulations using (3.25). This completes the proof of Theorem 2.2.

4 Proof of Theorem 2.3

To show (2.15), we note that equations (2.5) and (2.13) can be viewed as a Lax pair. Differentiating both sides of (2.13), together with (2.5), implies

$$N(\vec{n} + \vec{e}_l; x) \mathbf{P}_{\vec{n}+\vec{e}_l}(x) = (W'(\vec{n} + \vec{e}_l; x) + W(\vec{n} + \vec{e}_l; x) N(\vec{n}; x)) \mathbf{P}_{\vec{n}}(x),$$

for $l = 1, \dots, r$. Using (2.13) again, it follows

$$N(\vec{n} + \vec{e}_l; x) W(\vec{n} + \vec{e}_l; x) \mathbf{P}_{\vec{n}}(x) = (W'(\vec{n} + \vec{e}_l; x) + W(\vec{n} + \vec{e}_l; x) N(\vec{n}; x)) \mathbf{P}_{\vec{n}}(x). \quad (4.1)$$

It is now sufficient to show that the components of the vector $\mathbf{P}_{\vec{n}}$ are linearly independent whenever \vec{n} is a normal index. This assertion is already shown in [37]. For the convenience of the reader, we repeat the argument here. Suppose that (c_0, c_1, \dots, c_r) are such that

$$c_0 P_{\vec{n}} + \sum_{j=1}^r c_j P_{\vec{n}-\vec{e}_j} = 0,$$

then by comparing the leading coefficients, it follows $c_0 = 0$. If we multiply by x^{n_k-1} and integrate with respect to μ_k , then

$$c_k \int x^{n_k-1} P_{\vec{n}-\vec{e}_k}(x) d\mu_k(x) = 0.$$

We claim that the above integral does not vanish. Indeed, if the integral vanishes, then $P_{\vec{n}} - a P_{\vec{n}-\vec{e}_k}$ is a monic polynomial of degree $|\vec{n}|$ satisfying the orthogonality relations

(1.3) for any a , which contradicts the normality of \vec{n} . As a consequence, $c_k = 0$ for $k = 1, \dots, r$. The linear independence of the polynomials in $\mathbf{P}_{\vec{n}}$ and (4.1) gives the compatibility conditions (2.15). This completes the proof of Theorem 2.3.

Remark. From (2.13), it is readily seen that one can obtain the vector $\mathbf{P}_{\vec{n}+\vec{e}_i+\vec{e}_j}$ in two different ways. The first way is to first compute $\mathbf{P}_{\vec{n}+\vec{e}_i}$ from $\mathbf{P}_{\vec{n}}$ and then to compute $\mathbf{P}_{\vec{n}+\vec{e}_i+\vec{e}_j}$ from $\mathbf{P}_{\vec{n}+\vec{e}_i}$:

$$\mathbf{P}_{\vec{n}+\vec{e}_i+\vec{e}_j}(x) = W(\vec{n} + \vec{e}_i + \vec{e}_j; x)W(\vec{n} + \vec{e}_i; x)\mathbf{P}_{\vec{n}}(x).$$

The second way is to first compute $\mathbf{P}_{\vec{n}+\vec{e}_j}$ from $\mathbf{P}_{\vec{n}}$ and then to compute $\mathbf{P}_{\vec{n}+\vec{e}_i+\vec{e}_j}$ from $\mathbf{P}_{\vec{n}+\vec{e}_j}$:

$$\mathbf{P}_{\vec{n}+\vec{e}_i+\vec{e}_j}(x) = W(\vec{n} + \vec{e}_j + \vec{e}_i; x)W(\vec{n} + \vec{e}_j; x)\mathbf{P}_{\vec{n}}(x).$$

Hence, we obtain another kind of compatibility condition

$$W(\vec{n} + \vec{e}_i + \vec{e}_j; x)W(\vec{n} + \vec{e}_i; x) = W(\vec{n} + \vec{e}_j + \vec{e}_i; x)W(\vec{n} + \vec{e}_j; x).$$

This was observed in [37], and from the explicit formula of W the following nearest-neighbor recurrence relations for multiple orthogonal polynomials were obtained.

Theorem 4.1. [37, Theorem 3.2] *Suppose all multi-indices $\vec{n} \in \mathbb{N}^r$ are normal. Suppose $1 \leq i \neq j \leq r$, then the recurrence coefficients for the nearest neighbor recurrence relations (1.6) satisfy*

$$\begin{aligned} b_{\vec{n}+\vec{e}_i,j} - b_{\vec{n},j} &= b_{\vec{n}+\vec{e}_j,i} - b_{\vec{n},i}, \\ \sum_{k=1}^r a_{\vec{n}+\vec{e}_j,k} - \sum_{k=1}^r a_{\vec{n}+\vec{e}_i,k} &= \det \begin{pmatrix} b_{\vec{n}+\vec{e}_j,i} & b_{\vec{n},i} \\ b_{\vec{n}+\vec{e}_i,j} & b_{\vec{n},j} \end{pmatrix}, \\ \frac{a_{\vec{n},i}}{a_{\vec{n}+\vec{e}_j,i}} &= \frac{b_{\vec{n}-\vec{e}_i,j} - b_{\vec{n}-\vec{e}_i,i}}{b_{\vec{n},j} - b_{\vec{n},i}}. \end{aligned}$$

This theorem tells us that the recurrence coefficients of multiple orthogonal polynomials cannot be arbitrary, but satisfy a system of partial difference equations. This is not the case for the usual orthogonal polynomials.

5 Some examples

In this section, we shall derive the ladder equations and differential equations for several examples of multiple orthogonal polynomials. For convenience, it is assumed that $r = 2$ throughout this section. The (normal) multi-index \vec{n} is now given by $(n, m) \in \mathbb{N}^2$. We also use the following notation for the recurrence coefficients:

$$a_{\vec{n},1} = a_{n,m}, \quad a_{\vec{n},2} = b_{n,m}, \quad b_{\vec{n},1} = c_{n,m}, \quad b_{\vec{n},2} = d_{n,m}.$$

The recurrence relations (1.6) then read

$$xP_{n,m}(x) = P_{n+1,m}(x) + c_{n,m}P_{n,m}(x) + a_{n,m}P_{n-1,m}(x) + b_{n,m}P_{n,m-1}(x), \quad (5.1)$$

$$xP_{n,m}(x) = P_{n,m+1}(x) + d_{n,m}P_{n,m}(x) + a_{n,m}P_{n-1,m}(x) + b_{n,m}P_{n,m-1}(x), \quad (5.2)$$

with $a_{0,m} = 0$ and $b_{n,0} = 0$ for all $n, m \geq 0$. In view of Theorem 4.1, the following relations hold:

$$d_{n+1,m} - d_{n,m} = c_{n,m+1} - c_{n,m}, \quad (5.3)$$

$$b_{n+1,m} - b_{n,m+1} + a_{n+1,m} - a_{n,m+1} = \det \begin{pmatrix} d_{n+1,m} & d_{n,m} \\ c_{n,m+1} & c_{n,m} \end{pmatrix}, \quad (5.4)$$

$$\frac{a_{n,m+1}}{a_{n,m}} = \frac{c_{n,m} - d_{n,m}}{c_{n-1,m} - d_{n-1,m}}, \quad (5.5)$$

$$\frac{b_{n+1,m}}{b_{n,m}} = \frac{c_{n,m} - d_{n,m}}{c_{n,m-1} - d_{n,m-1}}. \quad (5.6)$$

5.1 Multiple Hermite polynomials

Multiple Hermite polynomials $H_{n,m}$ are type II multiple orthogonal polynomials defined by

$$\begin{aligned} \int_{-\infty}^{\infty} x^k H_{n,m}(x) e^{-x^2+c_1x} dx &= 0, & k = 0, 1, \dots, n-1, \\ \int_{-\infty}^{\infty} x^k H_{n,m}(x) e^{-x^2+c_2x} dx &= 0, & k = 0, 1, \dots, m-1, \end{aligned}$$

where $c_1 \neq c_2$; cf. [8], [28, §23.5] and [38, §3.4]. Hence, the functions $v_k(x)$ in Theorem 2.1 are equal to $x^2 - c_k x$. A simple calculation with the aid of (1.3) then gives

$$N(n, m; x) = \begin{pmatrix} 0 & 2a_{n,m} & 2b_{n,m} \\ -2 & 2x - c_1 & 0 \\ -2 & 0 & 2x - c_2 \end{pmatrix}. \quad (5.7)$$

Recall that $N(n, m; x)$ is defined by (2.5) and explicitly given in (2.4). This, together with (2.14) and the compatibility conditions (2.15), leads to the the following system of difference equations for the recurrence coefficients:

$$2a_{n,m} - 2a_{n+1,m} + 2b_{n,m} - 2b_{n+1,m} + 1 = 0, \quad (5.8)$$

$$2a_{n,m} - 2a_{n,m+1} + 2b_{n,m} - 2b_{n,m+1} + 1 = 0, \quad (5.9)$$

$$2c_{n,m} = c_1, \quad 2d_{n,m} = c_2, \quad (5.10)$$

$$2(d_{n,m-1} - c_{n,m-1}) - (c_2 - 2c_{n,m}) = 0, \quad (5.11)$$

$$2b_{n+1,m}(d_{n,m-1} - c_{n,m-1}) - b_{n,m}(c_2 - 2c_{n,m}) = 0, \quad (5.12)$$

$$2(d_{n-1,m} - c_{n-1,m}) + c_1 - 2d_{n,m} = 0, \quad (5.13)$$

$$2a_{n,m+1}(d_{n-1,m} - c_{n-1,m}) + a_{n,m}(c_1 - 2d_{n,m}) = 0. \quad (5.14)$$

It turns out one can easily obtain the recurrence coefficients explicitly from these equations. Indeed, from (5.10), it follows

$$c_{n,m} = c_1/2, \quad d_{n,m} = c_2/2.$$

Since $c_1 \neq c_2$, equations (5.11) and (5.12) imply that $b_{n,m}$ is independent of n , while (5.13) and (5.14) show that $a_{n,m}$ is independent of m . Hence, by (5.8) and (5.9), it follows

$$2a_{n,m} - 2a_{n,m+1} + 1 = 0, \quad 2b_{n,m} - 2b_{n+1,m} + 1 = 0. \quad (5.15)$$

Note that our initial conditions are $a_{0,m} = 0$ and $b_{n,0} = 0$, we then obtain from (5.15)

$$a_{n,m} = n/2, \quad b_{n,m} = m/2.$$

Thus, we arrive at the following lowering and raising equations for multiple Hermite polynomials $H_{n,m}$:

$$H'_{n,m}(x) = nH_{n-1,m}(x) + mH_{n,m-1}(x), \quad (5.16)$$

$$H'_{n-1,m}(x) = -2H_{n,m}(x) + (2x - c_1)H_{n-1,m}(x), \quad (5.17)$$

$$H'_{n,m-1}(x) = -2H_{n,m}(x) + (2x - c_2)H_{n,m-1}(x). \quad (5.18)$$

Observe that equation (5.17) (similarly (5.18)) can be alternatively written as

$$\left(e^{-x^2+c_1x} H_{n-1,m}(x) \right)' = -2e^{-x^2+c_1x} H_{n,m}(x).$$

These formulas are not new but can already be found in [28, §23.8.2].

Finally, by (5.7) and arguments at the end of Section 2.1, we see that the type II multiple orthogonal polynomial $H_{n,m}$ satisfies the following linear differential equation of order 3:

$$\begin{aligned} p'''(x) + (c_1 + c_2 - 4x)p''(x) + (c_1(c_2 - 2x) + 2(m + n - 1 - c_2x + 2x^2))p'(x) \\ + 2(c_1m + c_2n - 2(m + n)x)p(x) = 0. \end{aligned}$$

Similarly, we can derive a third order differential equation for the type I multiple Hermite polynomials $A_{(n,m),l}$, $l = 1, 2$, which is given by

$$\begin{aligned} q'''(x) - (c_1 + c_2 - 4x)q''(x) + (c_1(c_2 - 2x) + 2(m + n - 1 - c_2x + 2x^2))q'(x) \\ - 2(c_1m + c_2n - 2(m + n)x)q(x) = 0. \end{aligned}$$

We point out that the above differential equation is independent of l , which can also be seen from (2.9).

5.2 Multiple Laguerre polynomials of the second kind

These polynomials are defined by the orthogonality conditions

$$\begin{aligned} \int_0^\infty x^k L_{n,m}(x) x^\alpha e^{-c_1x} dx &= 0, & k = 0, 1, \dots, n-1, \\ \int_0^\infty x^k L_{n,m}(x) x^\alpha e^{-c_2x} dx &= 0, & k = 0, 1, \dots, m-1, \end{aligned}$$

where we assume that $\alpha > 0$ and $c_1, c_2 > 0$ with $c_1 \neq c_2$; cf. [8], [33, Remark 5 on p. 160], [28, §23.4.2] and [38, §3.3]. Hence, the functions $v_k(x)$ in Theorem 2.1 are equal to $-\alpha \ln x + c_k x$, $k = 1, 2$. An appeal to the biorthogonality conditions (1.4) implies that

$$N(n, m; x) = \frac{1}{x} \begin{pmatrix} -\alpha \int_0^\infty \frac{L_{n,m}(t) Q_{n,m}(t)}{t} dt & A(n, m) & B(n, m) \\ -\alpha \int_0^\infty \frac{L_{n-1,m}(t) Q_{n,m}(t)}{t} dt & C(n, m) - \alpha + c_1x & D(n, m) \\ -\alpha \int_0^\infty \frac{L_{n,m-1}(t) Q_{n,m}(t)}{t} dt & E(n, m) & F(n, m) - \alpha + c_2x \end{pmatrix}, \quad (5.19)$$

where $Q_{n,m}$ is defined in (1.5) with $w_j(x) = x^\alpha e^{-c_j x}$, $j = 1, 2$, and

$$A(n, m) = \alpha a_{n,m} \int_0^\infty \frac{L_{n,m}(t) Q_{n+1,m}(t)}{t} dt, \quad (5.20)$$

$$B(n, m) = \alpha b_{n,m} \int_0^\infty \frac{L_{n,m}(t) Q_{n,m+1}(t)}{t} dt, \quad (5.21)$$

$$C(n, m) = \alpha a_{n,m} \int_0^\infty \frac{L_{n-1,m}(t) Q_{n+1,m}(t)}{t} dt, \quad (5.22)$$

$$D(n, m) = \alpha b_{n,m} \int_0^\infty \frac{L_{n-1,m}(t) Q_{n,m+1}(t)}{t} dt, \quad (5.23)$$

$$E(n, m) = \alpha a_{n,m} \int_0^\infty \frac{L_{n,m-1}(t) Q_{n+1,m}(t)}{t} dt, \quad (5.24)$$

$$F(n, m) = \alpha b_{n,m} \int_0^\infty \frac{L_{n,m-1}(t) Q_{n,m+1}(t)}{t} dt, \quad (5.25)$$

are certain constants depending on n and m . We want to give an explicit representation of $N(n, m; x)$ in (5.19). We first observe that the first column in (5.19) can be evaluated by comparing the leading coefficients on both sides of the differential equations (2.1) and (2.2). For instance, by (2.1) and (5.19), it follows

$$xL'_{n,m}(x) = \left(-\alpha \int_0^\infty \frac{L_{n,m}(t) Q_{n,m}(t)}{t} dt \right) L_{n,m}(x) + A(n, m) L_{n-1,m}(x) + B(n, m) L_{n,m-1}(x).$$

Recall that $L_{n,m}(x) = x^{n+m} + \dots$, hence by comparing the coefficients of order $n+m$ on both sides of the above equation, it is readily seen that

$$\alpha \int_0^\infty \frac{L_{n,m}(t) Q_{n,m}(t)}{t} dt = -(n+m). \quad (5.26)$$

Similarly, we obtain

$$\alpha \int_0^\infty \frac{L_{n-1,m}(t) Q_{n,m}(t)}{t} dt = c_1, \quad \alpha \int_0^\infty \frac{L_{n,m-1}(t) Q_{n,m}(t)}{t} dt = c_2. \quad (5.27)$$

To estimate the integrals in (5.20)–(5.25), we shall make use of the compatibility conditions (2.15) and the known results of the recurrence coefficients:

$$a_{n,m} = \frac{(n+m+\alpha)n}{c_1^2}, \quad b_{n,m} = \frac{(n+m+\alpha)m}{c_2^2}, \quad (5.28)$$

and

$$c_{n,m} = \frac{2n+m+\alpha+1}{c_1} + \frac{m}{c_2}, \quad d_{n,m} = \frac{n+2m+\alpha+1}{c_2} + \frac{n}{c_1}; \quad (5.29)$$

see [37, §5.4]. The compatibility conditions (2.15) in this case are given by

$$\begin{aligned} xN(n+1, m; x) & \begin{pmatrix} x - c_{n,m} & -a_{n,m} & -b_{n,m} \\ 1 & 0 & 0 \\ 1 & 0 & d_{n,m-1} - c_{n,m-1} \end{pmatrix} \\ & = \begin{pmatrix} x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} x - c_{n,m} & -a_{n,m} & -b_{n,m} \\ 1 & 0 & 0 \\ 1 & 0 & d_{n,m-1} - c_{n,m-1} \end{pmatrix} xN(n, m; x), \end{aligned} \quad (5.30)$$

and

$$\begin{aligned}
& xN(n, m+1; x) \begin{pmatrix} x - d_{n,m} & -a_{n,m} & -b_{n,m} \\ 1 & c_{n-1,m} - d_{n-1,m} & 0 \\ 1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} x - d_{n,m} & -a_{n,m} & -b_{n,m} \\ 1 & c_{n-1,m} - d_{n-1,m} & 0 \\ 1 & 0 & 0 \end{pmatrix} xN(n, m; x). \quad (5.31)
\end{aligned}$$

With the aid of (5.19) and the explicit formulas (5.26)–(5.29), we see from the $(2, 2)$ -entry in (5.30) that

$$A(n, m) = c_1 a_{n,m} = \frac{(n + m + \alpha)n}{c_1}.$$

Similarly, the $(3, 3)$ -entry in (5.31) implies

$$B(n, m) = c_2 b_{n,m} = \frac{(n + m + \alpha)m}{c_2}.$$

Then, we observe from the $(2, 3)$ -entry in (5.30) that

$$c_1 b_{n,m} + (d_{n,m-1} - c_{n,m-1})D(n+1, m) = B(n, m),$$

which gives

$$D(n, m) = -\frac{mc_1}{c_2}.$$

Similarly, the $(3, 2)$ -entry in (5.31) implies that

$$E(n, m) = -\frac{nc_2}{c_1},$$

which is independent of m . Finally, we obtain from the $(2, 1)$ -entry in (5.30) that

$$c_1 c_{n,m} + C(n+1, m) - \alpha + D(n+1, m) = n + m,$$

hence,

$$C(n, m) = -n.$$

An appeal to the $(3, 1)$ -entry in (5.30) gives

$$c_2 c_{n,m} + E(n+1, m) + F(n+1, m) - \alpha = n + m + c_2(c_{n,m-1} - d_{n,m-1}),$$

thus,

$$F(n, m) = -m.$$

Combining all these results, we find the following lowering and raising equations for multiple Laguerre polynomials of the second kind $L_{n,m}$:

$$\begin{aligned}
xL'_{n,m}(x) &= (n + m)L_{n,m}(x) + \left(\frac{(n + m + \alpha)n}{c_1}\right)L_{n-1,m}(x) \\
&\quad + \left(\frac{(n + m + \alpha)m}{c_2}\right)L_{n,m-1}(x), \\
xL'_{n-1,m}(x) &= -c_1 L_{n,m}(x) - (n + \alpha - c_1 x)L_{n-1,m}(x) - \frac{mc_1}{c_2}L_{n,m-1}(x), \\
xL'_{n,m-1}(x) &= -c_2 L_{n,m}(x) - \frac{nc_2}{c_1}L_{n-1,m}(x) - (m + \alpha - c_2 x)L_{n,m-1}(x).
\end{aligned}$$

Furthermore, the third order differential equation satisfied by $L_{n,m}$ is given by

$$\begin{aligned} x^2 p'''(x) - (x^2(c_1 + c_2) - 2x(\alpha + 1)) p''(x) \\ + (x^2 c_1 c_2 - x[(c_1 + c_2)(\alpha + 1) - nc_1 - mc_2] + \alpha(\alpha + 1)) p'(x) \\ - (xc_1 c_2(n + m) - \alpha(nc_1 + mc_2)) p(x) = 0, \end{aligned}$$

which can also be found in [5, §4.3]. Similarly, it can be shown that the type I multiple Laguerre polynomials of the second kind satisfy the following differential equation:

$$\begin{aligned} x^2 q'''(x) + (x^2(c_1 + c_2) - 2x(\alpha - 1)) q''(x) \\ + (x^2 c_1 c_2 - x[(c_1 + c_2)(\alpha - 1) - nc_1 - mc_2] + \alpha(\alpha - 1)) q'(x) \\ + (xc_1 c_2(n + m) - \alpha(nc_1 + mc_2)) q(x) = 0. \end{aligned}$$

5.3 Multiple Laguerre polynomials of the first kind

These polynomials are defined by the orthogonality conditions

$$\begin{aligned} \int_0^\infty x^k L_{n,m}(x) x^{\alpha_1} e^{-x} dx &= 0, \quad k = 0, 1, \dots, n-1, \\ \int_0^\infty x^k L_{n,m}(x) x^{\alpha_2} e^{-x} dx &= 0, \quad k = 0, 1, \dots, m-1, \end{aligned}$$

where $\alpha_1, \alpha_2 > 0$ and $\alpha_1 - \alpha_2 \notin \mathbb{Z}$; cf. [8], [28, §23.4.1] and [38, §3.2]. The functions $v_k(x)$ in Theorem 2.1 are now equal to $-\alpha_k \ln x + x$, $k = 1, 2$. A straightforward calculation using (1.4) gives

$$N(n, m; x) = \frac{1}{x} \begin{pmatrix} -\int_0^\infty \frac{L_{n,m}(t) Q_{n,m}^*(t)}{t} dt & A^*(n, m) & B^*(n, m) \\ -\int_0^\infty \frac{L_{n-1,m}(t) Q_{n,m}^*(t)}{t} dt & C^*(n, m) - \alpha_1 + x & D^*(n, m) \\ -\int_0^\infty \frac{L_{n,m-1}(t) Q_{n,m}^*(t)}{t} dt & E^*(n, m) & F^*(n, m) - \alpha_2 + x \end{pmatrix},$$

where we define

$$Q_{n,m}^*(t) = (\alpha_1 A_{(n,m),1}(t) t^{\alpha_1} + \alpha_2 A_{(n,m),2}(t) t^{\alpha_2}) e^{-t},$$

with $A_{(n,m),j}$, $j = 1, 2$ the associated type I multiple orthogonal polynomials and

$$\begin{aligned} A^*(n, m) &= a_{n,m} \int_0^\infty \frac{L_{n,m}(t) Q_{n+1,m}^*(t)}{t} dt, \\ B^*(n, m) &= b_{n,m} \int_0^\infty \frac{L_{n,m}(t) Q_{n,m+1}^*(t)}{t} dt, \\ C^*(n, m) &= a_{n,m} \int_0^\infty \frac{L_{n-1,m}(t) Q_{n+1,m}^*(t)}{t} dt, \\ D^*(n, m) &= b_{n,m} \int_0^\infty \frac{L_{n-1,m}(t) Q_{n,m+1}^*(t)}{t} dt, \\ E^*(n, m) &= a_{n,m} \int_0^\infty \frac{L_{n,m-1}(t) Q_{n+1,m}^*(t)}{t} dt, \\ F^*(n, m) &= b_{n,m} \int_0^\infty \frac{L_{n,m-1}(t) Q_{n,m+1}^*(t)}{t} dt, \end{aligned}$$

are certain constants depending on n and m . Again by comparing the leading coefficients on both sides of the differential equations (2.1) and (2.2), we have

$$\int_0^\infty \frac{L_{n,m}(t)Q_{n,m}^*(t)}{t} dt = -(n+m)$$

$$\int_0^\infty \frac{L_{n-1,m}(t)Q_{n,m}^*(t)}{t} dt = \int_0^\infty \frac{L_{n,m-1}(t)Q_{n,m}^*(t)}{t} dt = 1.$$

Since the recurrence coefficients are explicitly given by

$$a_{n,m} = \frac{n(n+\alpha_1)(n+\alpha_1-\alpha_2)}{n-m+\alpha_1-\alpha_2}, \quad b_{n,m} = \frac{m(m+\alpha_2)(m+\alpha_2-\alpha_1)}{m-n+\alpha_2-\alpha_1},$$

$$c_{n,m} = 2n+m+\alpha_1+1, \quad d_{n,m} = n+2m+\alpha_2+1,$$

see [37, §5.3], the same strategy as in Section 5.2 gives

$$A^*(n,m) = \frac{n(n+\alpha_1)(n+\alpha_1-\alpha_2)}{n-m+\alpha_1-\alpha_2}, \quad B^*(n,m) = \frac{m(m+\alpha_2)(m+\alpha_2-\alpha_1)}{m-n+\alpha_2-\alpha_1},$$

$$C^*(n,m) = -n, \quad F^*(n,m) = -m,$$

$$D^*(n,m) = E^*(n,m) = 0.$$

Hence, we find the following ladder equations for multiple Laguerre polynomials of the first kind $L_{n,m}$:

$$xL'_{n,m}(x) = (n+m)L_{n,m}(x) + \left(\frac{n(n+\alpha_1)(n+\alpha_1-\alpha_2)}{n-m+\alpha_1-\alpha_2} \right) L_{n-1,m}(x)$$

$$+ \left(\frac{m(m+\alpha_2)(m+\alpha_2-\alpha_1)}{m-n+\alpha_2-\alpha_1} \right) L_{n,m-1}(x),$$

$$xL'_{n-1,m}(x) = -L_{n,m}(x) - (n+\alpha_1-x) L_{n-1,m}(x),$$

$$xL'_{n,m-1}(x) = -L_{n,m}(x) - (m+\alpha_2-x) L_{n,m-1}(x).$$

Furthermore, the third order differential equation satisfied by $L_{n,m}$ is given by

$$x^2 p'''(x) + (-2x^2 + (\alpha_1 + \alpha_2 + 3)x) p''(x)$$

$$+ (x^2 - x(\alpha_1 + \alpha_2 - n - m + 3) + (\alpha_1 + 1)(\alpha_2 + 1)) p'(x)$$

$$- (x(n+m) - (n+m+nm + \alpha_1 m + \alpha_2 n)) p(x) = 0,$$

which can also be found in [5, §4.3]. Similarly, we have that the type I multiple Laguerre polynomials of the first kind satisfy the following differential equation:

$$x^2 q'''(x) + (2x^2 - (\alpha_1 + \alpha_2 - 3)x) q''(x)$$

$$+ (x^2 - x(\alpha_1 + \alpha_2 - n - m - 3) + (\alpha_1 - 1)(\alpha_2 - 1)) q'(x)$$

$$+ (x(n+m) - (mn - n - m + \alpha_1 m + \alpha_2 n)) q(x) = 0.$$

5.4 Multiple exponential polynomials with cubic potentials

These polynomials are defined by the orthogonality conditions

$$\begin{aligned} \int_{\Gamma} x^k P_{n,m}(x) e^{-x^3/3 - c_1 x} dx &= 0, & k = 0, 1, \dots, n-1, \\ \int_{\Gamma} x^k P_{n,m}(x) e^{-x^3/3 - c_2 x} dx &= 0, & k = 0, 1, \dots, m-1, \end{aligned}$$

where $c_1 \neq c_2$ and the contour Γ is taken to be the set $\{x : \Re x^3 < 0\}$, or simply, $\{x : \arg x = \pm 2\pi/3\}$. This definition generalizes the orthogonal polynomials introduced by Magnus in [32, §6]. Since $v_k(x) = x^3/3 + c_k x$, we obtain from (1.4) and (1.6) (or (5.1) and (5.2) in case $r = 2$) that

$$N(n, m; x) = \begin{pmatrix} -a_{n,m} - b_{n,m} & a_{n,m}(x + c_{n,m}) & b_{n,m}(x + d_{n,m}) \\ -x - c_{n-1,m} & x^2 + a_{n,m} + c_1 & b_{n,m} \\ -x - d_{n,m-1} & a_{n,m} & x^2 + b_{n,m} + c_2 \end{pmatrix}.$$

If we introduce the matrix form for N :

$$N(n, m; x) = F_2(n, m)x^2 + F_1(n, m)x + F_0(n, m)$$

and similarly for W in (2.14):

$$W(n+1, m; x) = Rx + R_1, \quad W(n, m+1; x) = Rx + R_2,$$

where

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$R_1 = \begin{pmatrix} -c_{n,m} & -a_{n,m} & -b_{n,m} \\ 1 & 0 & 0 \\ 1 & 0 & d_{n,m-1} - c_{n,m-1} \end{pmatrix}, \quad R_2 = \begin{pmatrix} -d_{n,m} & -a_{n,m} & -b_{n,m} \\ 1 & c_{n-1,m} - d_{n-1,m} & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

then the compatibility conditions (2.15) are given by the following equations (which are not identically zero), after collecting the coefficients with respect to x :

$$\begin{aligned} F_1(n+1, m)R_1 + F_0(n+1, m)R &= RF_0(n, m) + R_1F_1(n, m), \\ F_1(n, m+1)R_2 + F_0(n, m+1)R &= RF_0(n, m) + R_2F_1(n, m), \\ F_0(n+1, m)R_1 &= R + R_1F_0(n, m), \\ F_0(n, m+1)R_2 &= R + R_2F_0(n, m). \end{aligned}$$

Although in this case we do not know the exact expressions of the recurrence coefficients, the above relations indeed give us many nonzero equations for these coefficients. It can be shown that all but the following four equations can be simplified using them and (5.3)

and (5.4):

$$c_1 + a_{n,m} + a_{n+1,m} + b_{n,m} + b_{n+1,m} + c_{n,m}^2 = 0, \quad (5.32)$$

$$c_2 + a_{n,m} + a_{n,m+1} + b_{n,m} + b_{n,m+1} + d_{n,m}^2 = 0, \quad (5.33)$$

$$\begin{aligned} & -1 - b_{n,m}c_{n,m} + b_{n+1,m}c_{n,m} - a_{n,m}(c_{n-1,m} + c_{n,m}) \\ & + a_{n+1,m}(c_{n,m} + c_{n+1,m}) - b_{n,m}d_{n,m-1} + b_{n+1,m}d_{n+1,m} = 0, \end{aligned} \quad (5.34)$$

$$\begin{aligned} & -1 - b_{n,m}d_{n,m-1} - b_{n,m}d_{n,m} - a_{n,m}(c_{n-1,m} + d_{n,m}) \\ & + a_{n,m+1}(c_{n,m+1} + d_{n,m}) + b_{n,m+1}d_{n,m+1} + b_{n,m+1}d_{n,m} = 0. \end{aligned} \quad (5.35)$$

In addition to equations (5.3)–(5.6) one can get, for instance,

$$\det \begin{pmatrix} d_{n+1,m} & d_{n,m} \\ c_{n,m+1} & c_{n,m} \end{pmatrix} = d_{n,m}^2 - c_{n,m}^2 - c_1 + c_2.$$

In particular, we see that equations (5.32) and (5.33) are nonlinear, which is different from the previous examples. We hope these relations will be helpful in the further study of multiple exponential polynomials with cubic potentials.

Finally, we point out that it is also possible to derive the differential equations for the associated type I and type II multiple orthogonal polynomials. Since they are cubersome, we shall not write them down here, but mention that they are of the form

$$a_2(x)p'''(x) + a_4(x)p''(x) + a_6(x)p'(x) + a_5(x)p(x) = 0,$$

where $a_k(x)$ are polynomials in x of degree k with the coefficients depending on the recurrence coefficients $a_{n,m}, b_{n,m}, c_{n,m}, d_{n,m}$.

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